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Abstract: Consider the Delaunay graph and the Voronoi tessellation constructed with respect to a planar Poisson point process. The sequence of nuclei of the Voronoi cells that are crossed by a line defines a path on the Delaunay graph. We show that the evolution of this path is governed by a Markov chain. We study the ergodic properties of the chain and find its stationary distribution. As a corollary we obtain the ratio of the mean path length to the Euclidean distance between the end points, and hence a bound for the mean asymptotic length of the shortest path.

We apply these results to define a family of simple incremental algorithms for constructing short paths on the Delaunay graph and discuss potential applications to routing in mobile communications networks.

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Chemins Markoviens sur le Graphe de Delaunay Associé à un Processus de Poisson

et applications au problème du routage dans les réseaux de stations mobiles

Résumé : Considérons la triangulation de Delaunay et le pavage de Voronoi associés à un processus ponctuel de Poisson dans le plan. La suite des centres des cellules de Voronoi qui sont traversées par une ligne droite définit un chemin sur la triangulation de Delaunay. Nous montrons que l'évolution de ce chemin est markovienne. Nous étudions les propriétés ergodiques de cette chaîne de Markov et nous donnons sa distribution stationnaire. S'en déduisent le rapport asymptotique entre la longueur du chemin et la distance parcourue sur la ligne droite, ainsi que des bornes sur les chemins les plus courts dans le graphe de Delaunay.

Nous utilisons ceci pour définir et analyser une famille d'algorithmes de routage dans les réseaux de communication qui ne nécessitent pas de connaissance de l'état complet du réseau. L'intérêt de ce type de routage apparaît clairement dans le cas de réseaux de stations mobiles.

Mots-clés : processus de Poisson, triangulation de Delaunay, diagramme de Voronoï, chemin le plus court, percolation, routage, réseaux mobiles

1 Introduction

The Voronoi tessellation and Delaunay triangulation are classical objects in many branches of applied mathematics: computational geometry, image analysis, networks (see Okabe et al. (1992) and references therein). They are defined as follows.

Let Π be an at most countable set of points in $\mathbb{R}^2$. A *Delaunay graph* constructed with respect to the vertex set $\Pi = \{Z_i, i \in \mathbb{N}\}$ connects all the pairs of points $\{Z_{i_1}, Z_{i_2}\}$ such that there exists a disc having Z_{i_1} and Z_{i_2} on its boundary and no points of Π in its interior. The Delaunay graph is unique whenever all the points of Π are in general position, namely no three points are co-linear, and no four points are co-circular. In this case the graph is planar and constitutes a triangulation of the plane.

The *Voronoi cell of nucleus* Z_i is a domain V_{Z_i} that consists of the points of the plane which are closer to Z_i than to any other $Z_j \in \Pi$. The collection of all Voronoi cells forms the *Voronoi tessellation* with respect to the set Π. The Delaunay graph and the Voronoi tessellation are dual: there is an edge between Z_i and Z_j in the Delaunay graph if and only if the cells V_{Z_i} and V_{Z_j} share an edge.

The main object of this paper are certain paths on the Delaunay graph. A *path* $p(Z_i, Z_j)$ between two nodes Z_i and Z_j of the Delaunay graph is a sequence of segments

$$[Z_i, Z_{i_1}], [Z_{i_1}, Z_{i_2}], \ldots, [Z_{i_{n-1}}, Z_j],$$

where each segment is an edge of the graph. The *length* $|p(Z_i, Z_j)|$ of the path is the sum of the lengths of all of its segments.

We say that a class of paths *t-approximates the Euclidean distance* (respectively, asymptotically t-approximates the Euclidean distance) if for each path $p(Z_i, Z_j)$ in this class

$$|p(Z_i, Z_j)| \leq t\|Z_i - Z_j\|; \tag{1}$$

$$\text{(respectively,} \quad \limsup_{\|Z_i - Z_j\| \to \infty} |p(Z_i, Z_j)|/\|Z_i - Z_j\| \leq t \text{)}, \tag{2}$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^2$. Such approximations of the Euclidean distance have several potential applications, some of which will be outlined in Section 4 below.

Let Π be a *homogeneous Poisson point process* of intensity 1 in $\mathbb{R}^2$. In this case the graphs and the paths constructed with respect to Π are *random closed sets.* The lengths of the paths are random variables, and we will speak of *t-approximation in mean* if (1) holds for the expectation of $|p|$, and of *asymptotic t-approximation* if (2) holds with probability one.

The aim of the present paper is to give a probabilistic analysis of a class of "short" paths on the Delaunay graph which approximate the Euclidean distance well, both asymptotically and in mean.

The paper is structured as follows. In Section 2 we introduce the class of paths and show that their segments form a Markov chain. We then study the asymptotic behavior of this chain, establish its convergence to the stationary regime and find its stationary distribution. As a corollary we show that the class of Markov paths $4/\pi$-approximates in mean and asymptotically the Euclidean distance. This and other other mean characteristics are obtained in Theorem 3. The proofs are relegated to Section 5. In Section 3 we describe two modifications of the Markov path that lead to shorter paths at the cost of additional complexity. Finally, in Section 4 we discuss possible applications to routing in mobile communication networks; we introduce a distributed routing algorithm based on a local view of the network and we use the analytical results to characterize its mean performance.

2 The Markov path algorithm

The following notation is used throughout the paper:

- For any Borel set B, $\Pi(B)$ denotes the number of points of Π in B;
- By $T(x,y)$ we denote the point at which the bisector of the segment $[x,y]$ crosses the abscissa axis l;
- By $B_r(x)$ we denote an open disc of radius r centered at x.

Let s and t be two points in $\mathbb{R}^2$. Let Z_i and Z_j be the two points of Π which are the closest to s and t, respectively. Consider the sequence $V_{Z_i}, V_{Z_{i_0}}, \ldots, V_{Z_j}$ of cells successively crossed by the segment $[s,t]$. The sequence of nuclei of these cells defines a path $\hat{p} = \hat{p}(s,t,\Pi)$ on the Delaunay graph from Z_i to Z_j (see Figure 1). This path is the main object of our study.

Another way to define the end-points of a random path is to add two fixed points s and t to the vertex set Π and consider the path $\hat{p}(s,t,\Pi')$ with $\Pi' = \Pi \cup \{s\} \cup \{t\}$. Relation (8) at the end of this section shows that both paths have the same asymptotic behavior.

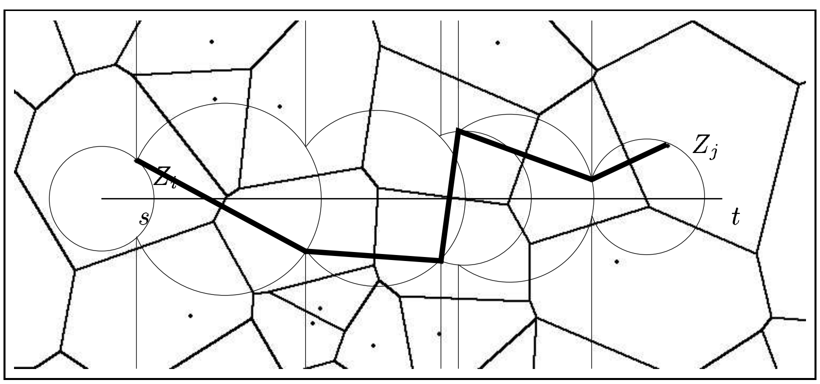

Figure 1: Markov path on the Poisson–Delaunay graph.

Remember that the underlying Poisson process Π is homogeneous and isotropic. Since we are interested in the distribution of $\hat{p}$, without loss of generality we may take $s = 0$ and let t belong to the positive abscissa axis l^+. The sequence of vertices of $\hat{p}(0,t,\Pi)$ is a finite subsequence of the infinite sequence $\{Z_{i_0}, Z_{i_1}, \dots\}$ of the nuclei of the cells crossed by l^+; this infinite path will be denoted $\hat{p}(0,\infty,\Pi)$.

Observe that if the disc $B_{\|Z_i - T(Z_i,Z_j)\|}(T(Z_i,Z_j))$ contains no point of Π, then

(i) the segment $[Z_i, Z_j]$ belongs to the Delaunay graph,

(ii) $T(Z_i,Z_j)$ is the point at which the abscissa axis l crosses the border between V_{Z_i} and V_{Z_j}.

The next proposition gives a criterion for an arbitrary sequence of points of Π to be the sequence of vertices of the path $\hat{p}(0,t,\Pi)$. Let $q_n = \{Z_{j_0}, Z_{j_1}, \dots, Z_{j_n}\}$ be an arbitrary sequence of points of Π. Denote $T_k = T(Z_{j_{k-1}}, Z_{j_k})$. Define the sets

$$\begin{aligned} B_0 &= B_{\|Z_{j_0}\|}(0);\\ B_k &= B_k(Z_{j_{k-1}}, Z_{j_k}) = B_{\|Z_{j_{k-1}} - T_k\|}(T_k), \quad k = 1,2,\dots,n;\\ D_k &= D_k(Z_{j_{k-2}}, Z_{j_{k-1}}) = \{x = (x^1,x^2) \in \mathbb{R}^2 : x^1 > Z^1_{j_{k-1}}\} \setminus B_{k-1}, \quad k = 1,2,\dots,n. \end{aligned}$$

Proposition 1. *A sequence q_n coincides with the n first vertices of the path $\hat{p}(0,\infty,\Pi)$ if and only if the following conditions are satisfied*

(i) $\Pi(B_k) = 0, \quad k = 0,1,\dots,n;$

(ii) $Z_{j_k} \in D_k, \quad k = 1,\dots,n.$

Note that if q_n coincides with $\hat{p}(0,t,\Pi)$ for some t, then the set $\{T_k\}$ is the restriction on the segment $[0,t]$ of the point process of cell border crossings by l.

A remarkable fact is that the sequence of segments $\{[Z_{i_{n-1}}, Z_{i_n}]\}_{n\geq 0}$ of the path $\hat{p}(0,\infty,\Pi)$ is a Markov chain. Here are some heuristic arguments (the proof is given in Section 5). Fix the history of the process up to the n-th step, that is up to $[Z_{n-1}, Z_n]$. From Proposition 1, Z_{n+1} is a point of Π in D_{n+1} such that $\Pi(B_{n+1}) = 0$. There exists a.s. exactly one such point because the points of Π are a.s. in a general position. Due to the strong Markov property of the Poisson process, the numbers of points of Π in the disjoint sets $B_{n+1} \cap D_{n+1}$ and $B_{n+1} \cap D^c_{n+1}$ are independent. From the history, it follows that $\Pi(B_{n+1} \cap D^c_{n+1}) = 0$. Therefore, the event $\Pi(B_{n+1}) = 0$ has the same conditional probability as the event

$$\{\Pi(B_{n+1} \cap D_{n+1}) = 0\} = \{\Pi(B_{n+1} \setminus B_n) = 0\}. \tag{3}$$

Note that the shape of D_{n+1} and the form of Condition (3) depend only on the pair (Z_{n-1}, Z_n), and not on the whole history of the process.

We now introduce a parameterization of $\hat{p}(0,\infty,\Pi)$ (see Figure 2). Let R_0, Φ_0 be the polar coordinates of Z_0. Denote by $\Phi_k \in (-\pi,\pi]$ the angle between $[T_k, Z_{i_k}]$ and $[T_k, T_{k+1}]$, and by $\Xi_i \in (-\pi,\pi]$ the angle between $[T_k, Z_{i_{k-1}}]$ and $[T_k, T_{k+1}]$. Put $R_k = \|T_k - Z_{i_{k-1}}\|$

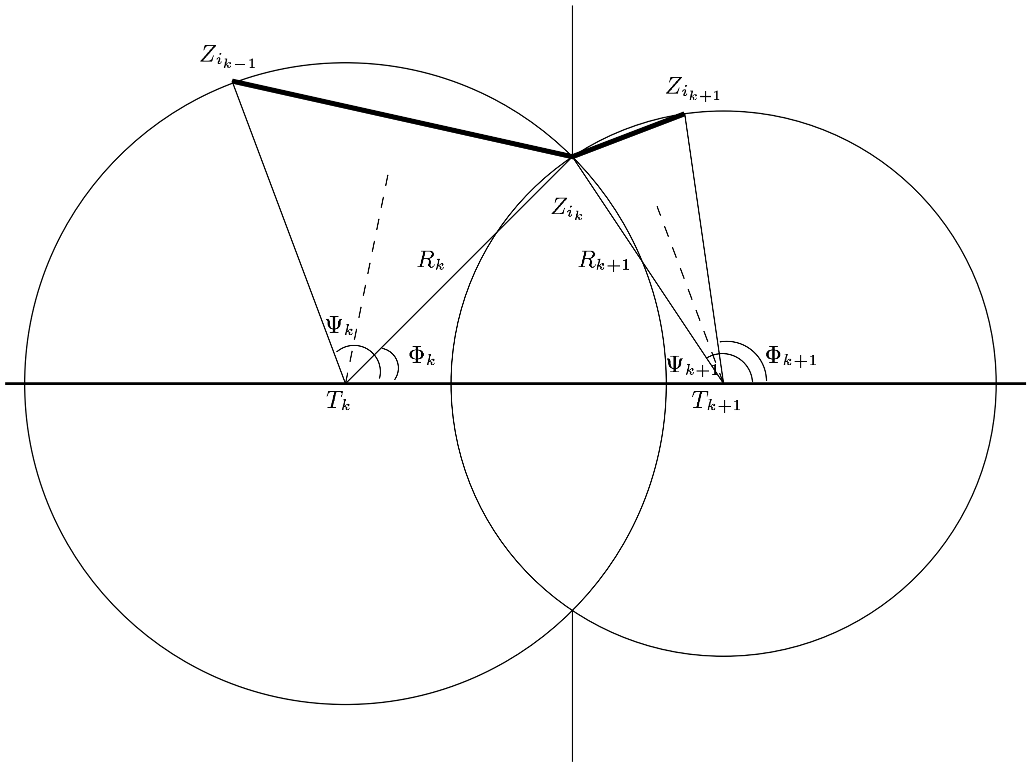

Figure 2: Two consecutive segments of the path $\hat{p}(0, t, \Pi)$

and $\Psi_k = |\Xi_k|$. The triple (R_k, Φ_k, Ξ_k) completely defines the length and the orientation of the segment $[Z_{i_{k-1}}, Z_{i_k}]$. Observe that if $Z_k \in D_k$, then $|\Phi_k| < \Psi_k$, and if $T_{k-1} < T_k$, then $|\Phi_{k-1}| < \Psi_k$.

Note the relation

$$R_{k+1} = \frac{R_k \sin|\Phi_k|}{\sin|\Xi_{k+1}|} = R_0 \prod_{j=0}^{k-1} \frac{\sin|\Phi_j|}{\sin \Psi_{j+1}}. \tag{4}$$

Since the sign of Ξ_k coincides with the sign of Φ_{k-1}, the first n segments of the path $\hat{p}(0, \infty, \Pi)$ are completely defined by the parameters

$$R_0, \Phi_0, \Phi_1, \Psi_1, \ldots, \Phi_n, \Psi_n\,. \tag{5}$$

Proposition 2. *The sequence* $\{(R_n, \Phi_n, \Psi_n)\}$ *is a Markov chain in the state space* $\mathsf{X} = \mathbb{R}_+ \times \Theta$, *where* $\Theta = \{(\phi, \psi) : 0 < |\phi| < \psi < \pi\}$. *The transition probabilities are given by* (12) *and* (13).

We study the limiting behavior of the chain by applying the Foster–Lyapunov criterion to test for various forms of stochastic stability. The criterion relies on the mean drift properties of the chain with respect to certain test functions and sets. The next theorem uses the standard formalism described, e. g., in Meyn and Tweedie (1993).

Theorem 1. *The Markov chain* $\{(R_n, \Phi_n, \Psi_n)\}$ *is regular, positive Harris recurrent, and* $\mathbf{V}$*-uniformly ergodic for the function* $\mathbf{V}$ *defined in* (23).

Various limit theorems hold for $\mathbf{V}$-uniformly ergodic chains; see, for example Theorem 17.0.1 in Meyn and Tweedie (1993).

To find the stationary distribution, consider the process $\mathcal{N} = \{T_n, n \in \mathbb{Z}\}$ of cell border crossings by the abscissa axis l. The parameterization introduced above extends naturally to the path $\hat{p}(0, -\infty, \Pi)$. The sequence of marks $\{M_n = (R_n, \Phi_n, \Psi_n), n \in \mathbb{Z}\}$ associated with the points of $\mathcal{N}$ coincides with the Markov chain $\{(R_n, \Phi_n, \Psi_n), n \geq 0\}$. Since the generating Poisson process Π is stationary, the marked point process $\{(T_n, M_n), n \in \mathbb{Z}\}$ is also stationary with respect to shifts along the abscissa axis. Therefore, the stationary distribution of the embedded Markov chain $\{(R_n, \Phi_n, \Psi_n), n \geq 0\}$ coincides with the Palm distribution $\mathbf{P}^0_{\mathcal{N}}$ of the process $\mathcal{N}$, which can be found using Palm calculus.

Theorem 2. *The stationary distribution* $\pi(\,\bullet\,)$ *of the Markov chain* $\{(R_n, \Phi_n, \Psi_n), n \geq 0\}$ *is given by*

$$\pi(\,\bullet\,) = \int_{\bullet} \frac{\pi}{2}(\cos\phi - \cos\psi) r^2 e^{-\pi r^2}\, d(r, \phi, \psi). \tag{6}$$

If $L : \mathsf{X} \to \mathbb{R}$ is a non-negative measurable function and $\lambda_{\mathcal{N}}$ is the intensity of the process $\mathcal{N}$, then by Campbell's formula

$$\mathbf{E} \sum_n L(M_k) \mathbb{1}_{T_k \in [0,t]} = \lambda_{\mathcal{N}} \|t\| \mathbf{E}^0_{\mathcal{N}} L(M_0) \tag{7}$$

(see, e. g., Baccelli and Brémaud (1994)). The path length is probably the most important of the mean path characteristics which can be obtained from this formula :

Theorem 3. *For each* $s, t \in \mathbb{R}^2$

$$\mathbf{E}|\hat{p}(s, t, \Pi)| = \frac{4}{\pi}\|s - t\|.$$

Simple scaling arguments show that the mean path length $\mathbf{E}|\hat{p}(s, t, \Pi)|$ does not depend on the intensity of the generating Poisson process. Some other mean characteristics of $\hat{p}(s, t, \Pi)$ can also be obtained from (7) like the number of segments, the inclination of a typical segment, etc.

As for the length of the path $\hat{p}(s, t, \Pi')$, defined in the beginning of this section, it can be shown that

$$\Big| \mathbf{E}|\hat{p}(s, t, \Pi)| - \mathbf{E}|\hat{p}(s, t, \Pi')| \Big| < C \tag{8}$$

for some constant C. See the proof of Theorem 3 in Section 5 for details.

3 Other short paths on the Delaunay graph

The shortest path Since each realization of Π is locally finite, for each pair of points Z_i, Z_j there exists a *shortest path* $p^*(Z_i, Z_j)$. We call the quantity $|p^*(Z_i, Z_j)|$ the *Delaunay distance* between Z_i and Z_j. In Keil and Gutwin (1992) it was shown that, for an arbitrary vertex set, the ratio *Delaunay distance/Euclidean distance* does not exceed $2\pi/(3\cos(\pi/6)) \approx 2.42$. On the other hand, the example constructed in Chew (1986) shows that this ratio can be arbitrarily close to $\pi/2 \approx 1.57$. Since then it is a standing conjecture that $\pi/2$ is indeed the *worst case*, i.e. there always exists a path that $\pi/2$-approximates the distance between its endpoints. However the examples of Delaunay graphs exhibiting such an extreme behavior seem rather artificial. This makes one believe that for more or less "regular" Delaunay graphs, the above ratio should be far less than 1.57. This claim is supported by the result of Theorem 3 which implies that for the Delaunay graph constructed with respect to the Poisson process Π, this ratio is asymptotically less than $4/\pi$.

Let s and t be two fixed points. Denote by $p^*(s,t,\Pi)$ the shortest path between the two vertices of Π which are the closest to s and t respectively. The family of paths $\{p^*(s,t,\Pi)\}_{s,t}$ is *subadditive*: for any three collinear points s, v, t such that $v \in [s,t]$

$$|p^*(s,t,\Pi)| \le |p^*(s,v,\Pi)| + |p^*(v,t,\Pi)|.$$

Therefore, by Kingman's subadditive theorem (see, e.g., Kingman (1973)), the finite limit

$$\kappa(p^*) = \lim_{\|t\|\to\infty} \|t\|^{-1} |p^*(0,t,\Pi)| \tag{9}$$

exists with probability one and in mean. In other words, the class of shortest paths on the Poisson–Delaunay graph asymptotically $\kappa(p^*)$-approximates the Euclidean distance.

Obviously, if some particular family of paths provides a κ-approximation to the Euclidean distance, then κ is an upper bound for $\kappa(p^*)$. The finite limit

$$\kappa(\hat{p}) = \lim_{\|t\|\to\infty} \|t\|^{-1} |\hat{p}(0,t,\Pi)|$$

exists with probability one. This follows from a direct application of the ergodic theorem: let $n(t)$ denote the number of points of $\{T_k\}$ in the segment $[0,t]$; we have

$$\lim_{\|t\|\to\infty} \|t\|^{-1} |\hat{p}(0,t,\Pi)| = \lim_{\|t\|\to\infty} \frac{n(t)}{\|t\|} \frac{1}{n(t)} \sum_{n=1}^{n(t)} \|Z_n - Z_{n-1}\|.$$

From Theorem 1, the Markov chain (Z_{n-1}, Z_n) is ergodic. Hence the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \|Z_n - Z_{n-1}\|$$

exists a.s. and is equal to $E^0_{\mathcal{N}} L_0$, where $L_0 = \|Z_1 - Z_0\|$. Similarly,

$$\lim_{\|t\|\to\infty} \frac{n(t)}{\|t\|} = \lambda_{\mathcal{N}} \quad a.\ s.$$

So, using results proved in § 5 (see (31)), we obtain that $\kappa(\hat{p}) = 4/\pi \approx 1.27$, and thus $\kappa(p^*) \leq 4/\pi$.

The value of $|\hat{p}|$ also facilitates the computation of the shortest path itself. Indeed, all the paths between Z_i and Z_j of length smaller than $|\hat{p}(Z_i, Z_j)|$ lie within the ellipse $\mathcal{E}$ defined by the focuses Z_i, Z_j, and the larger semi-axis $|\hat{p}(Z_i, Z_j)|/2$. Since Π is stationary and has intensity 1, the number of vertices of this graph has the order of the area of $\mathcal{E}$. For each realization, the number of points of Π in $\mathcal{E}$ is finite; hence, one of the algorithms for finding the shortest path on a finite graph can be applied (see, e. g., Gondran and Minoux (1979)).

First-passage percolation It is worth noting that the constant $\kappa(p^*)$ corresponds to the so-called *time constant* arising in first-passage percolation models. In these models a non-negative variable, the *passage time*, is associated with each edge of an infinite connected graph. The passage time along a path on the graph is the sum of the passage times of all the edges belonging to this path.

In many models, under appropriate conditions the limit (9) exists, and the shortest path length corresponds to the minimal passage time between the vertices of the graph which are the closest to s and t. If the vertex set is given by a Poisson process, and if the edge set includes the edges of the Delaunay triangulation, then the Markov path may be useful for obtaining an upper bound for the time constant.

As an illustration, consider the first-passage percolation model on the Poisson–Delaunay graph, where the passage times along the edges are independent random variables $\{\tau_i\}$ with common distribution function F. As shown in Vahidi-Asl and Wierman (1990), the time constant is finite in this model if and only if $\int_0^\infty [1 - F(t)]^3 dt < \infty$. Replacing in (7) the marks $L(M_k)$ by the passage times τ_k and using the fact that $\lambda_{\mathcal{N}} = 4\sqrt{\lambda}/\pi$ (see Stoyan et al. (1995, p. 331)), we immediately obtain that $\kappa(p^*) \leq 4\sqrt{\lambda}\mathbf{E}\tau_0/\pi$, where λ is the intensity of the underlying Poisson process.

The next example concerns the model introduced in Howard and Newman (1997). Let the passage time between any two points Z_i and Z_j of the Poisson process be equal to $\|Z_i - Z_j\|^\alpha$ for a fixed parameter $\alpha > 1$. Taking as mark $L(M_k)$ the length of an edge raised to the power α, we obtain from (31) that for this model

$$\kappa(p^*) \leq \frac{4\lambda^{\frac{1-\alpha}{2}}}{\pi} \left(\frac{2}{\sqrt{\pi}}\right)^\alpha \Gamma\left(\frac{\alpha+2}{2}\right) .$$

Modifications of the Markov path For applications, it is important that the paths can be built in an incremental way. Here is an algorithm for constructing the Markov path: start from Z_{i_0}, the point of Π which is the closest to s. Suppose that the path has been constructed to Z_{i_n}. If $t \notin V_{Z_{i_n}}$, then choose among the neighbors of $V_{Z_{i_n}}$ the next Voronoi cell that is crossed by the segment $[s, t]$. Take for $Z_{i_{n+1}}$ the nucleus of this cell.

Several other algorithms for constructing short paths on the Delaunay graph can be derived from the Markov path algorithm. We consider here two simple modifications:

1. Take as $Z_{i_{n+1}}$ the nucleus of the neighboring cell that is the closest to the destination t (see Figure 3.1).
2. Take as $Z_{i_{n+1}}$ the nucleus of the neighboring cell that is last crossed by $[s, t]$ (see Figure 3.2). For each realization of Π, the length of the path constructed in this way cannot exceed $|\hat{p}(s, t, \Pi)|$, because the set of vertices of this path is contained in the set of vertices of $\hat{p}(s, t, \Pi)$.

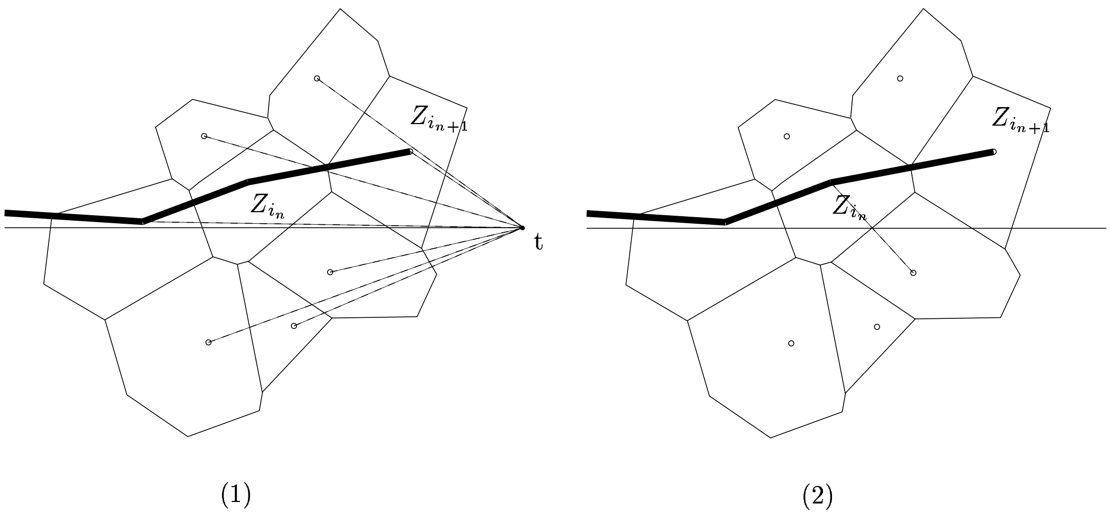

Figure 3: Modifications of the Markov Path Algorithm

Note that the paths constructed with these modified algorithms do not satisfy the subadditivity conditions of Kingman's theorem, and thus the limit (9) for these paths may not exist. Simulations nevertheless show that the asymptotic ratio *path length / Euclidean distance* is approximately 1.05 for the shortest path, 1.09 for the first modification, and 1.15 for the second.

4 Routing algorithms for mobile communication networks

Consider a mobile communication network that transfers packetized information using the mobile stations as intermediate relays. The packets between two stations are transmitted via communication links (e. g., radio channels).

A routing algorithm is a procedure for finding a path between two stations on the graph of links. The aim of this algorithm is usually to minimize the path length (or the number of intermediary hops). In networks where the positions of the stations are fixed, the shortest path to each destination can be pre-calculated and stored in a database (routing table) stored in each station. If the stations are mobile, the routing table requires frequent updates. Therefore it is desirable to have a routing algorithm that would work well with a rapidly changing graph of links and still yield reasonably short paths, without a total knowledge of the network.

Assume that each mobile station can determine its own location on the plane, and the locations of its closest neighbors, so that it can reconstruct its own Voronoi cell and Delaunay edges with respect to the point process of mobile stations at any time. Assume that each packet carries information on the locations of both the source and the destination end-stations, and that the network configuration changes that might occur during the period of transfer of a single packet can be neglected.

In this case, the Markov path algorithm (or one of its modifications) could be applied. The routing procedure for each mobile station can be described as follows:

1. The source sends a packet to the closest mobile routing station;
2. The mobile station receives a packet and extracts the positions of the source and the destination;
    - If the cell of the current station contains the destination the packet is relayed to it.
    - Else, the packet is relayed to the mobile station whose cell is crossed next by the source–destination line.

The advantage of this algorithm is that the decision on where to relay the packet is based only on the position of the closest neighbors, that is only on the local geometry of the network. Also note that if the mobile stations can be modeled by a Poisson point process, Theorem 3 gives explicitly the mean path length and the mean number of hops. The Markov property could also be used to determine variances or large deviations from this mean behavior.

5 Proofs

Proof of Proposition 1. Condition *(i)* for some $k \geq 1$ is equivalent to saying that the cells $V_{Z_{i_{k-1}}}$ and $V_{Z_{i_k}}$ are adjacent and that their common border is crossed by the abscissa axis at point T_k. For $k = 0$, this condition means that Z_{i_0} is the nucleus of the cell containing 0. Condition *(ii)* is satisfied if and only if $0 < T_0^1 < T_1^1 < \ldots < T_{n-1}^1$, which means that the cells are crossed in the same order as their nuclei appear in the sequence q_n. □

Proof of Proposition 2. We first determine the distribution of the first n segments of the path $\hat{p}(0, \infty, \Pi)$, defined by the random vector $\{Z_0, Z_1, \ldots, Z_n\} \in \mathbb{R}^{2(n+1)}$. Denote by $z_0^1, z_0^2, \ldots, z_n^1, z_n^2$ the coordinates of $z \in \mathbb{R}^{2(n+1)}$. From Proposition 1 it follows that the distribution of $\{Z_0, Z_1, \ldots, Z_n\}$ admits a density $d(z)$ with respect to the Lebesgue measure in $\mathbb{R}^{2(n+1)}$, with

$$d(z) = \exp\left\{-\lambda_2\left(\cup_{k=0}^{n} B_k(z)\right)\right\} \prod_{k=1}^{n} \mathbb{I}_{D_k(z)}((z_k^1, z_k^2)). \tag{10}$$

Here and in what follows, $\lambda_k(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^k$. Consider the mapping

$$A:\ y = (r_0, \phi_0, \phi_1, \psi_1, \dots, \phi_n, \psi_n) \mapsto z = (z_0^1, z_0^2, z_1^1, z_1^2, \dots, z_n^1, z_n^2),$$

defined by the following recurrence equations:

$$\begin{aligned}
z_0^1 &= r_0 \cos\phi_0 \\
z_0^2 &= r_0 \sin\phi_0 \\
z_{k+1}^1 &= z_k^1 + \frac{|z_k^2|(\cos\phi_{k+1} - \cos\psi_{k+1})}{\sin\psi_{k+1}}, \\
z_{k+1}^2 &= \frac{|z_k^2|\sin\phi_{k+1}}{\sin\psi_{k+1}}, \qquad k = 0, 1, \dots, n-1.
\end{aligned}$$

In Section 2 we introduced a parameterization of the path, such that

$$(Z_0, \dots, Z_n) = A(R_0, \Phi_0, \Phi_1, \Psi_1, \dots, \Phi_n, \Psi_n).$$

This mapping is a diffeomorphism between the two open subsets of $\mathbb{R}^{2(n+1)}$:

$$O = \{y \in \mathbb{R}^{2(n+1)} : r_0 > 0,\ |\phi_0| < \pi,\ |\phi_{k-1}| < \psi_k < |\phi_k| < \pi,\ k = 1, 2, \dots, n\}$$

and

$$O' = \{z \in \mathbb{R}^{2(n+1)} : (z_k^1, z_k^2) \in D_k(z),\ (z_k^1, z_k^2) \notin l,\ k = 1, 2, \dots, n\}.$$

Making in (10) the change of variables $z = A(y)$, we obtain the density

$$\bar{d}_n(r_0, \phi_0, \phi_1, \psi_1, \dots, \phi_n, \psi_n)$$

of the path parameters (5). The Jacobian of the mapping is equal to

$$\prod_{k=0}^{n} \bar{J}_k(r_0, \phi_0, \phi_1, \psi_1, \dots, \phi_k, \psi_k),$$

where

$$\begin{aligned}
\bar{J}_0 &= r_0 \\
\bar{J}_k(r_0, \phi_0, \phi_1, \psi_1, \dots, \phi_k, \psi_k) &= 2\left(r_0 \sin\phi_0 \prod_{j=1}^{k-1} \frac{\sin\phi_j}{\sin\psi_j}\right)^2 \frac{\cos\phi_k - \cos\psi_k}{\sin^3\psi_k}.
\end{aligned}$$

It is easy to verify that $\bar{J}_k \geq 0$ on the set O. Therefore,

$$\bar{d}_n(r_0, \phi_0, \phi_1, \psi_1, \ldots, \phi_n, \psi_n) \\ = \exp\{-\lambda_2 (\cup_{k=0}^n B_k)\} \prod_{k=0}^n \bar{J}_k(r_0, \phi_0, \phi_1, \psi_1, \ldots, \phi_i, \psi_i). \quad (11)$$

The conditional density $\bar{d}$ of (Φ_{n+1}, Ψ_{n+1}) at point (ϕ, ψ), given $\{R_0 = r_0, \Phi_0 = \phi_0, \Phi_k = \phi_k, \Psi_k = \psi_k,\ k = 1, 2, \ldots, n\}$ is equal to $\bar{d}_{n+1}/\bar{d}_n$. Since

$$\left(\cup_{k=0}^{n+1} B_k\right) \setminus (\cup_{k=0}^n B_k) = B_{n+1} \setminus B_n\,,$$

from (11) we obtain

$$\bar{d}(\phi, \psi \mid r_0, \phi_0, \phi_1, \psi_1 \ldots, \phi_n, \psi_n) \\ = \exp\{-\lambda_2 (B_{n+1} \setminus B_n)\} \bar{J}_{n+1}(r_0, \phi_0, \phi_1, \psi_1, \ldots, \phi_n, \psi_n, \phi, \psi).$$

To show that the process $\{(R_n, \Phi_n, \Psi_n)\}$ is a Markov chain, let us verify that the condition of the density $\bar{d}$ depends only on on r_n and ϕ_n. Indeed, for the first term

$$\lambda_2 (B_{n+1} \setminus B_n) = (r_n \sin\phi_n)^2 (v(\psi) - v(\phi_n)), \quad \text{where} \quad v(y) = \frac{|y| - \frac{1}{2}\sin|2y|}{\sin^2 y}.$$

For the second term, the relation (4) for R_n gives

$$\bar{J}_{n+1}(r_0, \phi_0, \phi_1, \psi_1, \ldots, \phi_n, \psi_n, \phi, \psi) = 2(r_n \sin\phi_n)^2 \frac{\cos\phi - \cos\psi}{\sin^3\psi}.$$

Hence $\{(R_n, \Phi_n, \Psi_n)\}$ is a Markov chain. Its one-step transition distribution is defined in the space X (see Proposition 2) by the following two components: the conditional density of $\{\Phi_{n+1}, \Psi_{n+1}\}$ given $\{R_n = r_n, \Phi_n = \phi_n\}$:

$$f(\phi, \psi \mid r_n, \phi_n) = \\ 2(r_n \sin\phi_n)^2 \frac{\cos\phi - \cos\psi}{\sin^3\psi} \exp\left\{-(r_n \sin\phi_n)^2 (v(\psi) - v(\phi_n))\right\} \mathbb{1}(\psi > |\phi_n|), \quad (12)$$

and the relation for R_{n+1}

$$R_{n+1} = \frac{r_n \sin|\phi_n|}{\sin\Psi_{n+1}}. \quad (13)$$

By Fubini's theorem, the conditional density of (Φ_{n+1}, Ψ_{n+1}) admits decomposition into a product of two conditional densities

$$f(\phi, \psi \mid r_n, \phi_n) = f_1(\phi \mid \psi)\, f_2(\psi \mid r_n, \phi_n), \quad (14)$$

where

$$f_1(\phi \mid \psi) = \frac{\cos\phi - \cos\psi}{2(\sin\psi - \psi\cos\psi)} \mathbb{I}(|\phi| < \psi), \tag{15}$$

$$f_2(\psi \mid r_n, \phi_n) = \frac{\partial F(r_n, \phi_n, \psi)}{\partial \psi} \mathbb{I}(|\phi_n| < \psi < \pi), \tag{16}$$

with

$$F(r_n, \phi_n, \psi) = -\exp\left\{-(r_n \sin\phi_n)^2 \left(v(\psi) - v(\phi_n)\right)\right\}. \tag{17}$$

This remark concludes the proof of Proposition 2. □

The proof of Theorem 1 is divided into several lemmas. Denote by $\mathcal{B}(\mathsf{X})$ the Borel σ-field on X. The next lemma shows that the chain is irreducible with respect to the Lebesgue measure on X.

Lemma 1. *For each $B \in \mathcal{B}(\mathsf{X})$ such that $\lambda_3(B) > 0$, the two-step transition probability $\mathbf{P}^2(u, B)$ considered as a function of $u = (r, \phi, \psi)$ is strictly positive.*

Proof. The two-step transition probability has the form

$$\mathbf{P}^2(u, B) = \int_{\Theta\times\Theta} \mathbb{I}_B(\{r_2, \phi_2, \psi_2\}) f(\phi_2, \psi_2 \mid r_1, \phi_1)\, f(\phi_1, \psi_1 \mid r, \phi)\, d\phi_2\, d\psi_2\, d\phi_1\, d\psi_1, \tag{18}$$

where f is the conditional density defined in (12), and where

$$r_1 = r\frac{\sin|\phi|}{\sin\psi_1}, \quad \text{and} \quad r_2 = \frac{r\, \sin|\phi|\, \sin|\phi_1|}{\sin\psi_1\, \sin\psi_2}. \tag{19}$$

For each $(r_2^0, \phi_2^0, \psi_2^0)$ in the interior of $B \in \mathcal{B}(\mathsf{X})$ and for each $u = (r, \phi, \psi)$ there exists $(\phi_1^0, \psi_1^0) \in \Theta$ such that

$$r_2^0 = \frac{r\, \sin|\phi|\, \sin|\phi_1^0|}{\sin\psi_1^0\, \sin\psi_2^0}$$

and in addition

$$\psi_1^0 > |\phi|; \qquad \psi_1^0 > |\phi_1^0|; \qquad |\phi_1^0| < \psi_2^0.$$

The indicator function in (18) equals 1 in some neighborhood of (ϕ_2^0, ψ_2^0). Also the product of the densities under the integral is positive in some neighborhood of $(\phi_1^0, \psi_1^0, \phi_2^0, \psi_2^0)$. Thus the integral in (18) is positive. □

Corollary 4. *The chain $\{(R_n, \Phi_n, \Psi_n)\}$ is ψ-irreducible. (see Meyn and Tweedie (1993), p. 89).*

Lemma 2. *The set*

$$C_M = \{u = (r,\phi,\psi) \in \mathsf{X} : r \le M \quad \text{or} \quad r > M, \quad r\sin|\phi| \le M, \quad |\phi| \le \pi/2\}$$

is ν_i-small for $i \ge 3$ (see the definition in Meyn and Tweedie (1993), p. 106). This means that for all $i \ge 3$, there exists a non-trivial measure ν_i on $\mathcal{B}(\mathsf{X})$ such that for each $B \in \mathcal{B}(\mathsf{X})$,

$$\inf_{u\in C_M} \mathbf{P}^i(u,B) \ge \nu_i(B), \quad i \ge 3.$$

Proof. For each $B^0 \in \mathcal{B}(\mathsf{X})$ and $i \ge 3$, the following inequality holds

$$\inf_{u\in C_M} \mathbf{P}^i(u,B) \ge \inf_{u\in C_M} \mathbf{P}(u,B^0) \left(\inf_{u\in B^0} \mathbf{P}(u,B^0) \right)^{i-3} \inf_{u\in B^0} \mathbf{P}^2(u,B). \tag{20}$$

Consider the set

$$B^0 = \{u_1 = (r_1,\phi_1,\psi_1) \in \mathsf{X} : \phi_1 \in [\pi/11, \pi/10],\, r_1 \in [2M, 3M],\, \pi/2 \le \psi_1 < \pi\}.$$

Let us first show that

$$\inf_{u\in C_M} \mathbf{P}(u,B^0) \ge \delta > 0. \tag{21}$$

As $B^0 \subset C_M$, this will also imply that

$$\inf_{u\in B^0} \mathbf{P}(u,B^0) \ge \delta.$$

Assume that $u = (r,\phi,\psi) \in C_M$. From (15), when $\psi_1 > \pi/2$ we have

$$\int_{[\pi/11,\pi/10]} f_1(\phi_1 \mid \psi_1)\, d\phi_1 > \delta_1 > 0.$$

From decomposition (14), we get

$$\mathbf{P}(u,B^0) > \delta_1 \int_{\max\{\pi/2,|\phi|\}}^{\pi} \mathbb{I}_{[2M,3M]}(r_1) f_2(\psi_1 \mid r,\phi)\, d\psi_1. \tag{22}$$

Put

$$\alpha_1 = \pi - \arcsin\frac{r\sin|\phi|}{2M}, \quad \alpha_2 = \pi - \arcsin\frac{r\sin|\phi|}{3M}.$$

When $(r,\phi,\psi) \in C_M$,

$$\max\{\pi/2, |\phi|\} \le \alpha_1 < \alpha_2 \le \pi.$$

Since $r_1 = r \sin|\phi| / \sin\psi_1$, the integrand in (22) differs from zero only within the interval $\psi_1 \in [\alpha_1, \alpha_2]$. Thus, the integral in (22) is equal to $\mathbf{P}\{\Psi_{n+1} \in [\alpha_1, \alpha_2] \bigm| R_n = r, \Phi_n = \phi\}$. Using (16) and (17) it can be shown that

$$\mathbf{P}\Big\{\Psi_{n+1} \in [\alpha_1, \alpha_2] \Bigm| R_n = r, \Phi_n = \phi\Big\} \geq \delta_2 > 0.$$

From here follows (21) with $\delta = \delta_1 \delta_2$.

The distribution $\mathbf{P}^2(u, B)$ admits a density $\gamma(u, v)$ with respect to the Lebesgue measure in X. It can be constructed in the same way as in the proof of Proposition 2. It can be shown that $\gamma(u, v) > \delta_3$ for some $\delta_3 > 0$ when $u \in B^0, v \in B^0$. Now it follows from inequality (20) that the minorization measures

$$\nu_i(B) = \delta^{i-2} \delta_3 \int_{B \cap B^0} du, \qquad i \geq 3.$$

satisfy the condition of the Lemma. □

Corollary 5. *The Markov chain $\{(R_n, \Phi_n, \Psi_n)\}$ is aperiodic.*

Let $\mathbf{V} : \mathsf{X} \to \mathbb{R}_+$ be a measurable function. Denote

$$\mathbf{PV}(r, \phi, \psi) = \mathbf{E}\Big[\mathbf{V}(R_{n+1}, \Phi_{n+1}, \Psi_{n+1}) \Bigm| R_n = r, \Phi_n = \phi, \Psi_n = \psi\Big].$$

Define the mean drift operator Δ as follows

$$\Delta\mathbf{V}(r, \phi, \psi) = \mathbf{PV}(r, \phi, \psi) - \mathbf{V}(r, \phi, \psi).$$

Lemma 3 (Geometrical drift towards C_M). *There exists a function $\mathbf{V} : \mathsf{X} \to [1, \infty)$ and constants β, $M > 0$, and $b < \infty$ such that*

$$\Delta\mathbf{V}(u) \leq -\beta\mathbf{V}(u) + b\mathbb{1}_{C_M}(u), \quad u = (r, \phi, \psi) \in \mathsf{X}.$$

Proof. The proof is carried out with the function

$$\mathbf{V}(u) \equiv \mathbf{V}(r, \phi, \psi) = \max\{\, r \sin|\phi/2|\,, 1\}. \tag{23}$$

Write

$$\begin{aligned} \mathbf{PV}(r, \phi, \psi) &\leq \mathbf{E}\Big[R_{n+1} \sin|\Phi_{n+1}/2| \Bigm| R_n = r, \Phi_n = \phi\Big] + 1 \\ &= \mathbf{E}\Big[R_{n+1}\, \mathbf{E}\Big[\sin|\Phi_{n+1}/2| \Bigm| \Psi_{n+1}\Big] \Bigm| R_n = r, \Phi_n = \phi\Big] + 1. \end{aligned}$$

Using the conditional density in (15) it can be easily verified that

$$\mathbf{E}\Big[\sin|\Phi_{n+1}/2| \Bigm| \Psi_{n+1} = \psi_1\Big] < \psi_1/5.$$

Therefore,

$$\begin{aligned}\mathbf{PV}(r,\phi,\psi) \leq & \mathbf{E}\left[\frac{R_{n+1}\Psi_{n+1}}{5} \,\Big|\, R_n = r, \Phi_n = \phi\right] + 1 \\ = & \int_{|\phi|}^{\pi} \frac{r \sin|\phi|}{\sin\psi_1} \frac{\psi_1}{5\sin\psi_1} \frac{\partial F(r,\phi,\psi_1)}{\partial\psi_1}\, d\psi_1 + 1.\end{aligned} \tag{24}$$

First, suppose that $(r,\phi,\psi) \in C_M$. Let us show that $\mathbf{PV}(r,\phi,\psi)$ is bounded by a constant. If $r \leq M$ and $|\phi| > \pi/2$, then from (24) we have

$$\begin{aligned}\mathbf{PV}(r,\phi,\psi) \leq & \frac{\pi}{5}\mathbf{E}\left[R_{n+1} \,\Big|\, R_n = r, \Phi_n = \phi\right] + 1 \\ \leq & \frac{\pi}{5}\left(2r + \mathbf{E}\left[R_{n+1}\, \mathbb{1}_{R_{n+1} > 2R_n} \,\Big|\, R_n = r, \Phi_n = \phi\right]\right) + 1 \\ \leq & \frac{2\pi M}{5} + \frac{\pi}{5}\int_{2r}^{\infty} \mathbf{P}\left(R_{n+1} > x \,\Big|\, R_n = r, \Phi_n = \phi\right) dx + 1.\end{aligned} \tag{25}$$

From the geometrical viewpoint the inequalities $|\phi| > \pi/2$ and $R_{n+1} > x > 2R_n$ imply that no points of the Poisson process Π lie in the domain $B_{n+1} \setminus B_n$, and that the surface of this domain is greater than the surface of a disc of radius $x/2$. Hence, the probability in the last integral of (25) does not exceed $\exp(-x^2/4)$, and so

$$\mathbf{PV}(r,\phi,\psi) \leq \frac{\pi}{5}(2M+1) + 1, \qquad (r,\phi,\psi) \in C_M, \quad |\phi| > \pi/2. \tag{26}$$

Now let $|\phi| \leq \pi/2$. Since

$$\frac{\psi_1}{\sin\psi_1} \leq \frac{\pi}{2}, \quad \psi \in (0, \pi/2],$$

we have

$$\begin{aligned}\mathbf{PV}(r,\phi,\psi) \leq & \frac{\pi r \sin|\phi|}{10} + \pi\mathbf{E}\left[R_{n+1}\, \mathbb{1}_{\Psi_{n+1} > \pi/2,\, R_{n+1} \leq M\sqrt{2}} \,\Big|\, R_n = r, \Phi_n = \phi\right] \\ & + \pi\mathbf{E}\left[R_{n+1}\, \mathbb{1}_{\Psi_{n+1} > \pi/2,\, R_{n+1} > M\sqrt{2}} \,\Big|\, R_n = r, \Phi_n = \phi\right] + 1 \\ \leq & \frac{\pi M}{10} + \pi M\sqrt{2} + \pi\int_{M\sqrt{2}}^{\infty} \mathbf{P}\left(R_{n+1} > x \,\Big|\, R_n = r, \Phi_n = \phi\right) dx + 1.\end{aligned}$$

Similar geometrical arguments apply in this case, showing that

$$\mathbf{PV}(r,\phi,\psi) \leq \pi\left(M/10 + M\sqrt{2} + 1\right) + 1, \qquad (r,\phi,\psi) \in C_M, \quad |\phi| \leq \pi/2. \tag{27}$$

Secondly, suppose that $u \in C_M^c$. Integrating (24) by parts, we get

$$\mathbf{PV}(r,\phi,\psi) \leq \frac{r\phi}{5} - \int_{|\phi|}^{\pi} \frac{r\sin|\phi|}{5}\left(\frac{\psi_1}{\sin\psi_1}\right)' F(r,\phi,\psi_1)\, d\psi_1 + 1.$$

Then

$$\Delta\mathbf{V}(r,\phi,\psi) \leq r\sin|\phi/2|\left(\frac{|\phi|}{5\sin|\phi/2|}-1\right)+I(r,\phi), \tag{28}$$

where

$$I(r,\phi) = -\int_{|\phi|}^{\pi}\frac{r\sin|\phi|}{5}\left(\frac{\psi_1}{\sin\psi_1}\right)' F(r,\phi,\psi_1)\,d\psi_1+1.$$

Using (13), (16), and (17) it can be shown that

$$I(r,\phi) = \mathbf{E}\left[\frac{1}{10R_{n+1}} \mid R_n = r, \Phi_n = \phi\right]+1.$$

As follows from (13),

$$R_{n+1} \geq \begin{cases} R_n & \text{if } |\phi| \leq \pi/2, \\ R_n\sin\Phi_n & \text{else.} \end{cases}$$

Therefore,

$$I(r,\phi) \leq \frac{1}{10M}+1.$$

Since in (28)

$$\left(\frac{|\phi|}{5\sin|\phi/2|}-1\right) \leq -\frac{3}{10},$$

we have

$$\Delta\mathbf{V}(r,\phi,\psi) \leq -\frac{3}{10}r\sin|\phi/2|+\frac{1}{10M}+1.$$

Put $M = 20$. Then, for all $(r,\phi,\psi) \in C_M^c$,

$$\mathbf{V}(r,\phi,\psi) = r\sin|\phi/2| \geq \frac{M}{2} \geq 10,$$

and therefore,

$$\Delta\mathbf{V}(r,\phi,\psi) \leq -\frac{1}{10}\mathbf{V}(r,\phi,\psi), \qquad (r,\phi,\psi) \in C_M^c. \tag{29}$$

From (26), (27), and (29) follows the statement of the lemma with

$$M = 20, \quad \beta = -\frac{1}{10}, \quad b = 100.$$

□

Proof of Theorem 1. The function $\mathbf{V}$ defined in Lemma 3 is unbounded off small (and hence, petite) sets (Meyn and Tweedie (1993), p. 191), i. e. its level sets

$$C_{\mathbf{V}}(K) = \{u \in \mathsf{X} : \mathbf{V}(u) \leq K\}, \qquad K > 0$$

are small. Indeed, any subset of a small set is small, and each set $C_{\mathbf{V}}(K)$ is contained in some $C_{M(K)}$ defined in Lemma 2. On the other hand, on C_M, the function $\mathbf{V}$ is bounded.

The assertion of Theorem 1 results from different modifications of the Foster–Lyapunov criteria. By Corollary 4 the chain $\{(R_n, \Phi_n, \Psi_n)\}$ is ψ-irreducible. By Lemma 2, the condition of a geometrical (and hence uniform) drift towards a petite set is satisfied. Moreover, the test function $\mathbf{V}$ is bounded on this set and finite everywhere. Therefore the chain $\{(R_n, \Phi_n, \Psi_n)\}$ is Harris recurrent and admits an invariant probability measure (see Meyn and Tweedie (1993) Theorem 11.3.4). The regularity of the chain follows from Theorem 11.3.15 in Meyn and Tweedie (1993). In addition to these properties, $\{(R_n, \Phi_n, \Psi_n)\}$ is aperiodic. This implies $\mathbf{V}$-uniform ergodicity (see Meyn and Tweedie (1993), Theorem 16.1.2) and geometric rate of convergence of the transition probabilities $\mathbf{P}^n(u, \cdot)$ to the invariant probability measure in the metric induced by $\mathbf{V}$ (see Meyn and Tweedie (1993)). □

Proof of Theorem 2. Denote by θ_t the shift of the probability space on which the process $\mathcal{N}$ is defined, so that $\theta_t \mathcal{N} = \{T_n - t\}_{n \in \mathbb{N}}$. It can be shown that the Palm probability is invariant with respect to the shift θ_{T_n} (see, e. g., Baccelli and Brémaud (1994), p. 18), and therefore,

$$\mathbf{P}^0_{\mathcal{N}}(M_n \in \cdot) = \mathbf{P}^0_{\theta_{T_1}\mathcal{N}}(M_n \in \cdot) = \mathbf{P}^0_{\mathcal{N}}(M_{n+1} \in \cdot).$$

From the other hand, if $\mathbf{P}(u, \cdot)$ is the transition probability of the Markov chain $\{M_n = (R_n, \Phi_n, \Psi_n)\}$, then

$$\mathbf{P}^0_{\mathcal{N}}(M_{n+1} \in \cdot) = \int_{\mathsf{X}} \mathbf{P}(M_{n+1} \in \cdot \bigm| M_n = (r, \phi, \psi)) \mathbf{P}^0_{\mathcal{N}}(M_n \in d(r, \phi, \psi)).$$

Therefore, $\mathbf{P}^0_{\mathcal{N}}$ is the stationary distribution of the chain $\{(R_n, \Phi_n, \Psi_n)\}$ (the uniqueness follows from Theorem 1).

To find the explicit form of the Palm probability, we use its local interpretation (see, e. g., Baccelli and Brémaud (1994), p. 39). For each $B \in \mathcal{B}(\mathsf{X})$,

$$\mathbf{P}^0_{\mathcal{N}}(M_1 \in B) = \lim_{h \downarrow 0} \mathbf{P}(M_1 \in B \bigm| T_1 \leq h).$$

Making the change of variables

$$(Z^1_0, Z^2_0, Z^1_1, Z^2_1) \to (T_1, R_1, \Phi_1, \Psi_1),$$

and acting as in proof of Proposition 2, we obtain the density of $(T_1, R_1, \Phi_1, \Psi_1)$

$$c(t, r, \phi, \psi) = r^2 (\cos\phi - \cos\psi) \exp\{-V(t, r, \psi)\} \, \mathbb{I}\left((t, r, \phi, \psi) \in \mathbb{R}_+ \times \mathsf{X}\right).$$

Here $V(t,r,\psi)$ is the surface of the domain $B_0 \cup B_1$. From this

$$\mathbf{P}^0_{\mathcal{N}}(M_1 \in B) = \lim_{h \downarrow 0} \frac{\int_{\mathsf{X}} \int_0^h \mathbb{1}_B(r,\phi,\psi)\, c(t_1,r,\phi,\psi)\, dt_1\, d(r,\phi,\psi)}{\int_{\mathsf{X}} \int_0^h c(t_1,r,\phi,\psi)\, dt_1\, d(r,\phi,\psi)}. \tag{30}$$

Since in $\mathbb{R}_+ \times \mathsf{X}$

$$c(t_1,r,\phi,\psi) \le c(0,r,\phi,\psi) = r^2(\cos\phi - \cos\psi)\exp\left\{-\pi r^2\right\},$$

by the Dominated Convergence Theorem, the limit in (30) is equal to

$$\mathbf{P}^0_{\mathcal{N}}(M_1 \in B) = \frac{\int_{\mathsf{X}} \mathbb{1}_B(r,\phi,\psi) c(0,r,\phi,\psi)\, d(r,\phi,\psi)}{\int_{\mathsf{X}} c(0,r,\phi,\psi)\, d(r,\phi,\psi)}.$$

Normalizing $c(0,r,\phi,\psi)$ we obtain the density of the Palm distribution. □

Proof of Theorem 3. The intensity $\lambda_{\mathcal{N}}$ can be computed as the expectation of T_1 with respect to the Palm distribution of the process $\mathcal{N}$. As it was mentioned in Section 3, $\lambda_{\mathcal{N}} = 4/\pi$. The path length is obtained directly from (6) and (7) with

$$L(M_k) = \|Z_{k-1} - Z_k\| = 2R_k \sin\frac{\Psi_k - \Phi_k}{2}.$$

If, more generally,

$$L(M_k) = \|Z_{k-1} - Z_k\|^\alpha, \quad \alpha > -1,$$

then

$$\mathbf{E}\sum_n L(M_k)\mathbb{1}_{T_k \in [0,t]} = \frac{4\|t\|}{\pi}\left(\frac{2}{\sqrt{\pi}}\right)^\alpha \Gamma\left(\frac{\alpha+2}{2}\right). \tag{31}$$

Finally, let us estimate the difference of length between $\hat{p}(s,t,\Pi)$ and $\hat{p}(s,t,\Pi')$. We will show that with large probability, the two path merge when defined in the same probability space. Denote

$$B(x,y) = B_{\|x-T(x,y)\|}(T(x,y))$$

and

$$\begin{aligned}
B_1(s,t,Z) &= B_{\|s-Z\|}(s) \cup B_{\|t-Z\|}(t) \cup B(s,t)\,,\\
B_2(s,t,Z,Z') &= B_{\|s-Z\|}(s) \cup B_{\|t-Z'\|}(t) \cup B(s,t)\,,\\
B_3(s,t,Z,Z') &= B(s,Z) \cup B(t,Z')\,,\\
B_4(s,t,Z) &= B(s,Z) \cup B(t,Z)\,.
\end{aligned}$$

Consider the following events, which form a partition of the probability space:

$$
\begin{aligned}
E_1 &= \{\exists\, Z \in \Pi : \Pi(B_1(s,t,Z)) = 0\}\,,\\
E_2 &= \{\exists\, Z, Z' \in \Pi : \Pi(B_2(s,t,Z,Z')) = 0\}\,,\\
E_3 &= \{\exists\, Z, Z' \in \Pi \cap B(s,t) : \Pi(B_3(s,t,Z,Z')) = 0\}\,,\\
E_4 &= \{\exists\, Z \in \Pi \cap B(s,t) : \Pi(B_4(s,t,Z)) = 0\}\,.
\end{aligned}
$$

The events E_1 and E_2 imply that $|\hat{p}(s,t,\Pi)| = \|s-t\|$. On E_1, s and t belong to V_Z, and $|\hat{p}(s,t,\Pi)| = 0$. On E_2, the path $\hat{p}(s,t,\Pi)$ lies in the closure of the disc of radius

$$
r(s,t,Z,Z') = \|s-Z\| + \frac{3}{2}\|s-t\| + \|t-Z'\|,
$$

centered in $T(s,t)$. Event E_3 implies that the points Z and Z' belong both to $\hat{p}(s,t,\Pi)$ and $\hat{p}(s,t,\Pi')$, which means that between these two points, the two paths coincide. Moreover, the remaining two parts of $\hat{p}(s,t,\Pi)$ lie in the closures of $B_{\|s-Z\|}(s)$ and $B_{\|t-Z'\|}(t)$, respectively, whereas the two remaining parts of $\hat{p}(s,t,\Pi')$ are just two segments of lengths $\|s-Z\|$ and $\|t-Z'\|$. Now, to estimate the expectation of

$$
D(s,t) = \Big|\mathbf{E}|\hat{p}(s,t,\Pi)| - \mathbf{E}|\hat{p}(s,t,\Pi')|\Big|,
$$

we make use of the total probability formula and the following fact: if some path p on the Delaunay graph lies in a closed disc of radius r, then

$$
\mathbf{E}|p| \le 2\pi r^3.
$$

Therefore,

$$
\begin{aligned}
\mathbf{E}\left[D(s,t)\;\Big|\;E_3, Z = z, Z' = z'\right] &\le C(s,t,z,z')\\
&\equiv 2\pi\|z-s\|^3 + \|z-s\| + 2\pi\|z'-t\|^3 + \|z'-t\|. \qquad (32)
\end{aligned}
$$

Similarly, in the case E_4, when the common path consists of a single point

$$
\mathbf{E}\left[D(s,t)\;\Big|\;E_4, Z = z\right] \le C(s,t,z,z). \qquad (33)
$$

The point Z from E_1 has a density $f(z \mid E_1)$. In Cartesian coordinates

$$
f(z\,|\,E_1)\,\mathbf{P}(E_1) = \exp\{-\lambda_2\left(B_1(s,t,z)\right)\}\,\mathbb{1}_{\mathbb{R}^2\setminus B(s,t)}(z). \qquad (34)
$$

Similar densities exist in the cases E_2–E_4. Summarizing the above results, we have

$$\begin{aligned}
\mathbf{E}D(s,t) \quad \leq & \int \|s-t\| \exp\{-\lambda_2 (B_1(s,t,z))\}\, dz \\
+ & \int 2\pi r^3(s,t,z,z') \exp\{-\lambda_2 (B_2(s,t,z,z'))\}\, dz\, dz' \\
+ & \int C(s,t,z,z') \exp\{-\lambda_2 (B_3(s,t,z,z'))\}\, dz\, dz' \\
+ & \int C(s,t,z,z) \exp\{-\lambda_2 (B_4(s,t,z))\}\, dz. \qquad (35)
\end{aligned}$$

The first two integrals vanish as $\|s-t\|$ becomes large. The second two are bounded by constants. Hence, $\mathbf{E}D(s,t)$ is bounded by a constant. □

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